

# Bigravity from gradient expansion

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**Abstract.** We discuss how the ghost-free bigravity coupled with a single scalar field can be derived from a braneworld setup. We consider DGP two-brane model without radion stabilization. The bulk configuration is solved for given boundary metrics, and it is substituted back into the action to obtain the effective four-dimensional action. In order to obtain the ghost-free bigravity, we consider the gradient expansion in which the brane separation is supposed to be sufficiently small so that two boundary metrics are almost identical. The obtained effective theory is shown to be ghost free as expected, however, the interaction between two gravitons takes the Fierz-Pauli form at the leading order of the gradient expansion, even though we do not use the approximation of linear perturbation. We also find that the radion remains as a scalar field in the four-dimensional effective theory, but its coupling to the metrics is non-trivial.

**Keywords:** modified gravity, extra dimensions, gravity

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## 1 Introduction

In general, bigravity, *i.e.*, the gravitational model that contains two gravitons interacting with each other, was known to be suffered from an unavoidable ghost mode, which is called Boulware-Deser (BD) ghost [1]. Recently, however, the restriction of the interaction to the specific form of non-derivative coupling proposed by de Rham, Gabadadze and Tolley (dRGT) in massive gravity [2, 3] is found to evade the BD ghost problem also in the ghost-free bigravity [4–6]. This ghost-free bigravity allows us to investigate many applications of bigravity to building a cosmological model, which revealed that the ghost-free bigravity can be consistent with the observation of the current universe [7–13], although there exists some constraints on the model construction to avoid the gradient instability in cosmological perturbation at the early epoch [8, 11, 12, 14]. As a challenge to the viability of the model there are also discussions about the superluminality and acausality pointed out in the context of massive gravity [15–18], or the stability of the tuning of the ghost-free interaction against radiative corrections [19, 20].

The interaction between two metrics different from the dRGT form must be suppressed to realize a viable model, but we do not know the mechanism that realizes such fine tuning. In order to find a clue to this fine-tuning problem and also to give a UV completion of the ghost-free bigravity model, we attempted to derive the ghost-free bigravity as a low-energy effective theory starting with a healthy braneworld setup, which is naturally expected to be free from BD ghost [21]. In order to derive bigravity from the braneworld setup, we have to realize the mass hierarchy between the lowest massive Kaluza-Klein (KK) mode and the other KK modes. We considered Dvali-Gabadadze-Porrati (DGP) 2-brane model [22], because the Einstein-Hilbert terms localized on the branes effectively trap graviton modes on the respective branes, and as a consequence only the lowest and the second lowest masses of KK towers of gravitons, which correspond to these two localized modes, can be made to be light by tuning the contributions of the brane localized Einstein-Hilbert terms to be large compared with the bulk one [21, 23]. This tuning is achieved by setting  $\Delta y/r_c^{(\pm)} \ll 1$ , where  $\Delta y$  is the brane separation and  $r_c^{(\pm)} := M_{pl}^{(\pm)2}/2M_5^3$  with  $M_{pl}^{(\pm)}$  and  $M_5$  being the 4-dimensional Planck masses for the brane localized Einstein-Hilbert terms and 5-dimensional

Planck mass, respectively. An alternative way to lead to bigravity by discretizing the extra-dimension has been investigated in Ref. [24].

In this paper, we do not introduce any radion stabilization mechanism which fixes the brane separation, and hence there exists a scalar radion corresponding to the degree of freedom of the brane separation. Namely, the low-energy effective action should consist of two gravitons and one scalar radion. This model is even simpler to analyze because we do not have to employ the stabilization mechanism. Nevertheless, since the radion couples to both induced metrics on branes, there is a possibility that the low-energy effective theory derived from DGP 2-brane model may naturally give a non-trivial ghost-free doubly-coupled matter model different from the previously proposed ones [25–31]. Here, we should recall that DGP 2-brane model has two branches: the self-accelerating branch inevitably has either the Higuchi ghost or the radion ghost, while the normal branch is free from ghost when de Sitter universe is investigated [32, 33]. In order to get the ghost-free branch, we need to set the model parameters to satisfy  $1 - 2r_c^{(\pm)}(n_{(\pm)}^\mu \partial_\mu \log a) > 0$  on the respective branes [21], where  $a$  is the background bulk warp factor and  $n_{(\pm)}^\mu$  is the unit normal vector pointing toward the bulk on each brane. If we assume that the values of the extrinsic curvature on both branes are approximately the same, *i.e.*,  $K \approx K_+ \approx K_- \approx \mp 4(n_{(\pm)}^\mu \partial_\mu \log a)$ , the above condition requires  $|K| \lesssim 1/r_c^{(\pm)}$ . Combining the condition  $\Delta y/r_c^{(\pm)} \ll 1$ , the region of the model parameters of our interest is restricted to

$$|K\Delta y| \ll 1. \quad (1.1)$$

Since our interest is in the regime with small  $K\Delta y$ , we use the gradient expansion [34] to construct the bulk solution. We can systematically expand the action with respect to the small quantity,  $K\Delta y$ . Since the higher-order terms in the gradient expansion cannot cancel the lower-order ones, we naively expect that the action at each order is kept free from ghost. In this paper we consider the lowest order truncation of the gradient expansion. In order to obtain the action of the low-energy effective theory written in terms of the metrics on the  $(\pm)$ -branes,  $g_{\mu\nu}^{(\pm)}$ , we solve the bulk equations for given boundary metrics  $g_{\mu\nu}^{(\pm)}$  at the lowest order of the gradient expansion, and then we integrate out the bulk degrees of freedom by substituting back the obtained bulk solution into the action and performing the integration along the extra dimension. Our method using the gradient expansion will not always give a ghost-free effective action even if we start with a healthy higher dimensional model because the higher order corrections in the gradient expansion would naturally derive higher-derivative terms in the effective action which seem to introduce extra degrees of freedom including ghost modes in addition to the expected ones, which are two graviton modes and one scalar radion in the present case. We should interpret that the appearance of such extra degrees of freedom is responsible for the other bulk degrees of freedom with higher masses, which are in the present setup the massive KK gravitons with the mass squared of  $O(\Delta y^{-1})$  or higher. Therefore, the appearance of extra degrees of freedom itself will not directly indicate the breakdown of the gradient expansion. Although it would be unsatisfactory if such higher derivative terms remain even in the final result of the low-energy effective action, they should be understood to be treated by means of the method of order reduction, then.

Besides the fine-tuning problem of the ghost-free interaction form, the extension of the ghost-free bigravity to braneworld setup may solve the issue of gradient instability in cosmology and acausality, since both of them seem to be absent in the braneworld setup. The main difference between bigravity and the brane model will originate from the presence

of the extra massive graviton modes in the braneworld setup, which are inactive at low energies but will play a crucial role at high energies. Also, we expect that the stability against radiative corrections might be understood well from the viewpoint of the braneworld setup. However, our study presented in this paper is not matured yet to address these issues.

This paper is organized as follows: In Sec. II, we introduce the braneworld model that we consider in this paper and explain the strategy to obtain the effective action. In Sec. III, we start with the derivation of the effective action to the quadratic order in perturbation around a de Sitter brane background solution, and show that the obtained effective action contains the expected two gravitons and one scalar radion. In Sec. IV, we analyze the equation of motion derived from the effective action obtained in Sec. III with source energy momentum tensor and investigate the positivity of the action when we integrate out all the gravitational field degrees of freedom. In Sec. V we extend the method to obtain the effective action to the nonlinear level. In Sec. VI, we summarize the result and discuss possible future extension of the current work.

## 2 Model

We consider DGP two-brane model without the radion stabilization, whose action is written as

$$S = S_b + \sum_{\sigma=\pm} S_{\sigma}, \quad (2.1)$$

where

$$S_b := \frac{1}{2} \frac{M_{pl}^2}{2r_c^{(+)}} \int d^5x \sqrt{-{}^5g} \left( {}^5R - \frac{12}{\ell_\Lambda^2} \right), \quad (2.2)$$

$$S_+ := \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g_{(+)}} (R_{(+)} - 2\sigma_+), \quad (2.3)$$

$$S_- := \frac{\chi M_{pl}^2}{2} \int d^4x \sqrt{-g_{(-)}} (R_{(-)} - 2\sigma_-), \quad (2.4)$$

where  ${}^5g_{\mu\nu}$ ,  ${}^5R$ ,  $g_{\mu\nu}^{(\pm)}$ ,  $R_{(\pm)}$  are the 5-dimensional metric, 5-dimensional Ricci scalar, 4-dimensional metrics and Ricci scalars induced on  $(\pm)$ -branes, respectively.  $M_{pl}$  is the 4-dimensional Planck mass on  $(+)$ -brane, while  $M_{pl}^2/2r_c^{(+)}$  and  $\chi M_{pl}^2$  are the 5-dimensional bulk Planck mass cubed and 4-dimensional Planck mass squared on the  $(-)$ -brane, respectively.  $\sigma_{\pm}$  are the 4-dimensional cosmological constants on the respective branes, while  $6/\ell_\Lambda^2$  is the 5-dimensional cosmological constant. The equation of motion in the bulk spacetime is given as

$${}^5R_{\mu\nu} - \frac{1}{2} {}^5R {}^5g_{\mu\nu} + \frac{6}{\ell_\Lambda^2} g_{\mu\nu} = 0. \quad (2.5)$$

Here, we assume the  $Z_2$  symmetry across each brane. We adopt  $S_1/Z_2$  orbifold identification where the left hand side and right hand side with respect to each brane are identical. Then, the extrinsic curvatures on the  $(\pm)$ -branes,  $K_{\mu\nu}^{(\pm)}$ , are determined by the Israël's junction conditions:

$$K_{\mu\nu}^{(\pm)} - K^{(\pm)} g_{\mu\nu}^{(\pm)} = \pm r_c^{(\pm)} \left( G_{\mu\nu}^{(\pm)} + \sigma_{\pm} g_{\mu\nu}^{(\pm)} \right), \quad (2.6)$$

where  $r_c^{(-)} := r_c^{(+)}\chi$ .

We will derive the low-energy effective action from (2.1), by solving the bulk metric for given boundary metrics on the branes and integrating out the bulk degrees of freedom. In this paper, we employ the gradient expansion to find the bulk metric, assuming that the brane separation is sufficiently small compared with the 5-dimensional curvature scale, and hence the metrics on the both branes are identical at the leading order of the small separation expansion. Here, we describe the strategy to obtain the effective action in more detail. At the lowest order of the expansion, we can set the following ansatz:

$$ds^2 = N^2(x)dy^2 + g_{\mu\nu}dx^\mu dx^\nu, \quad (2.7)$$

while the  $(\pm)$ -branes locate at  $y = y^\pm$ , respectively. We use the gauge degrees of freedom to choose the lapse  $N$  and the shift  $N^\mu$  in the above form. Namely, we imposed  $\partial_y N = 0$  and  $N^\mu = 0$ . For given boundary geometries specified by  $g_{\mu\nu}^{(+)}(x)dx^\mu dx^\nu$  and  $\tilde{g}_{\mu\nu}(\tilde{x})d\tilde{x}^\mu d\tilde{x}^\nu$  ( $g_{\mu\nu}^{(-)}(x)$  is reserved for later use) on the respective branes, we should be able to find an appropriately interpolating bulk solution by solving the bulk equation of motion. Decomposing the bulk equations (2.5) with respect to the  $y = \text{constant}$  hypersurfaces under the above gauge fixing condition (2.7), we obtain

$$\partial_{\tilde{y}} K_{\mu\nu} = -2K_\mu^\rho K_{\rho\nu} + K K_{\mu\nu} + \frac{4}{\ell_\Lambda^2} g_{\mu\nu} - R_{\mu\nu} + \frac{N_{;\mu\nu}}{N}, \quad (2.8)$$

$$K_{\mu\nu} = -\frac{1}{2}\partial_{\tilde{y}} g_{\mu\nu}, \quad (2.9)$$

$$K^2 - K_\nu^\mu K_\mu^\nu = -\frac{12}{\ell_\Lambda^2} + R, \quad (2.10)$$

$$K_{;\mu} - K_{\mu;\nu}^\nu = 0, \quad (2.11)$$

where we define  $\partial_{\tilde{y}} \equiv \frac{1}{N}\partial_y$ .  $K_{\mu\nu}$  and  $R_{\mu\nu}$  are the extrinsic curvature and the 4-dimensional Ricci tensor evaluated on each  $y$ -constant surface, respectively. Here, we stress that  $g_{\mu\nu}$  and  $K_{\mu\nu}$  are defined not on the  $\tilde{y} \equiv Ny = \text{constant}$  hypersurface, but on the  $y = \text{constant}$  one.  $f_{;\mu}$  is the covariant derivative of the field  $f$  associated with the 4-dimensional metric  $g_{\mu\nu}$  (We reserve  $\nabla_\mu$  for later use). The indices are raised and lowered also using  $g_{\mu\nu}$ .

Once we fix the way how to relate the 4-dimensional coordinates on one brane with those on the other by the gauge choice of the bulk metric, the coordinates on the other brane is specified. If we adopt the above bulk metric ansatz (2.7), the way of connecting between two branes is completely specified. In these specific coordinates, the metric is given by

$$g_{\mu\nu}^{(-)}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{g}_{\alpha\beta}(\tilde{x}), \quad (2.12)$$

and  $g_{\mu\nu}^{(-)}(x)$  cannot be chosen arbitrarily. If we solve Eqs. (2.8)-(2.10) (excluding (2.11)) with the boundary metrics  $g_{\mu\nu}^{(+)}$  and  $g_{\mu\nu}^{(-)}$ , the coordinate transformation  $\tilde{x}^\alpha(x)$  is constrained by the momentum constraint (2.11). Using the gradient expansion, we can expand  $g_{\mu\nu}$  around the middle point  $\bar{y} = 0$  in the bulk as

$$g_{\mu\nu}^{(\pm)} = \bar{g}_{\mu\nu} + \overline{\partial_{\tilde{y}} g_{\mu\nu}} \tilde{y}^\pm + \frac{1}{2} \overline{\partial_{\tilde{y}}^2 g_{\mu\nu}} (\tilde{y}^\pm)^2 + \dots, \quad (2.13)$$

where  $\bar{g}_{\mu\nu} := g_{\mu\nu}(0)$ , and  $\tilde{y}^\pm \equiv N(x)y^\pm$ . As we set  $\bar{y} = 0$ , the values of  $\tilde{y}$  on the respective branes are related with each other as  $\tilde{y}^- = -\tilde{y}^+$ . Since the positions of the  $(\pm)$ -branes in  $y$

are fixed at  $\mp y_0$ , respectively, we have

$$\tilde{y}^+ = -Ny_0 < 0, \quad \tilde{y}^- = Ny_0 > 0. \quad (2.14)$$

Using the expression (2.13) and the evolution equations (2.8) and (2.9), we can obtain  $\bar{g}_{\mu\nu}$  and  $\bar{K}_{\mu\nu}$  in terms of  $g_{\mu\nu}^{(\pm)}$  and  $\tilde{y}^+$ , and then the Hamiltonian constraint (2.10) determines  $\tilde{y}^+$  and hence  $N$  in terms of  $g_{\mu\nu}^{(\pm)}$ . The effective action written in terms of  $g_{\mu\nu}^{(\pm)}$  should be given by substituting the bulk solution into the action

$$S = \frac{M_{pl}^2}{2r_c^{(+)} } \oint d^5x \sqrt{-g} \left( R + K^2 - K_\nu^\mu K_\mu^\nu - \frac{12}{\ell_\Lambda^2} \right) \\ + \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g_{(+)}} (R_{(+)} - 2\sigma_+) + \frac{\chi M_{pl}^2}{2} \int d^4x \sqrt{-g_{(-)}} (R_{(-)} - 2\sigma_-), \quad (2.15)$$

where the 5-dimensional Einstein-Hilbert action is also decomposed into the 4+1 form with respect to the  $y = \text{constant}$  hypersurfaces. Since the integrand of Eq. (2.15) does not include any second derivative of the metric in the  $y$ -direction, the boundary Gibbons-Hawking terms on the branes are unnecessary in this expression. The integral along  $y$ -direction reduces to integrals of simple powers of  $y$  by expanding the integrand at  $y = 0$  with the aid of the gradient expansion. We do not manifestly impose the momentum constraints, because they are automatically imposed later. As long as the extrinsic curvatures evaluated on the branes,  $K_{\mu\nu}^{(\pm)}$ , for the obtained bulk solution, agree with the ones that are derived from the variation of the effective action with respect to  $g_{\mu\nu}^{(\pm)}$ , as is expected, we obtain the junction conditions (2.6) from the variation of the effective action. By taking the divergence of the junction conditions (2.6), the right hand side identically vanishes, and hence we find that the momentum constraints (2.11) are imposed.

### 3 Perturbation around de Sitter spacetime

First, we consider the perturbation around a de Sitter brane solution with the comoving curvature  $H$  and calculate the effective action quadratic in the perturbation, as a warm-up. The background spacetime is given by

$$ds^2 = dy^2 + a^2(y) \gamma_{\mu\nu} dx^\mu dx^\nu, \quad (3.1)$$

where we set  $N = 1$ ,  $a(y = 0) = 1$  and  $\gamma_{\mu\nu}$  is the 4-dimensional de Sitter metric with the expansion rate  $H$ . Then the background solution is determined by the Hamiltonian constraint (2.10) and the junction condition (2.6):

$$\mathcal{H} := \frac{\partial_y a}{a} = \pm \sqrt{-\frac{1}{\ell_\Lambda^2} + \frac{H^2}{a^2}}, \quad (3.2)$$

$$\mp \mathcal{H}_\pm = r_c^{(\pm)} \left( -\frac{\sigma_\pm}{3} + \frac{H^2}{a_\pm^2} \right), \quad (3.3)$$

where  $\mathcal{H}_\pm := \mathcal{H}(y = \mp y_0)$  and  $a_\pm := a(y = \mp y_0)$ . As we choose the convention that  $\mathcal{H}(y = 0) > 0$ , the (+)-brane should have a positive tension ( $\sigma_+ > 0$ ). Then, Eq. (3.2) is solved as

$$\mathcal{H} = -\frac{1}{\ell_\Lambda} \tan \left( \frac{y}{\ell_\Lambda} + A \right), \quad (3.4)$$

which is the background value of  $-K/4$ , with the integration constant

$$A = -\arccos\left(\frac{1}{\ell_\Lambda H}\right) (< 0) . \quad (3.5)$$

Then we obtain

$$a = B \cos\left(\frac{y}{\ell_\Lambda} + A\right) , \quad (3.6)$$

with the integration constant

$$B = \ell_\Lambda H . \quad (3.7)$$

Using the above expression, we find

$$\mathcal{H}_\pm := \mathcal{H}(\mp y_0) = \frac{1}{\tilde{\ell}_\Lambda} \pm H^2 y_0 + \frac{1}{\tilde{\ell}_\Lambda} \mathcal{O}(H^2 y_0^2) . \quad (3.8)$$

where we define  $\tilde{\ell}_\Lambda$  by

$$\tilde{\ell}_\Lambda^{-1} := \mathcal{H}(y=0) = \sqrt{-\ell_\Lambda^{-2} + H^2} . \quad (3.9)$$

We represent the perturbation around this background as

$$g_{\mu\nu}^{(\pm)} = a^2(\mp y_0) \left( \gamma_{\mu\nu} + h_{\mu\nu}^{(\pm)} \right) , \quad (3.10)$$

$$N = 1 + x , \quad i.e. , \quad \tilde{y}^\pm = \mp y_0(1 + x) . \quad (3.11)$$

Here we note that the value of  $y$ -coordinate at the location of the perturbed (+)-brane remains to be  $y = -y_0$  in our coordinate system. To make the calculation easier, we define new variables as

$$\mathcal{G}_{\mu\nu}(y) := a^{-2}(y) g_{\mu\nu}(y) , \quad (3.12)$$

$$\mathcal{K}_{\mu\nu}(y) := a^{-2}(y) \left( K_{\mu\nu}(y) + \frac{\mathcal{H}(y)}{N} g_{\mu\nu}(y) \right) , \quad (3.13)$$

so as to isolate the  $y$ -dependence related to the background warp factor  $a(y)$ . Then, with these new variables, Eqs. (2.8)-(2.11) become

$$\begin{aligned} \partial_{\tilde{y}} \mathcal{K}_{\mu\nu} &= -2\mathcal{K}_\mu^\rho \mathcal{K}_{\rho\nu} + \mathcal{K} \mathcal{K}_{\mu\nu} - \frac{\mathcal{H}}{N} \mathcal{K} \mathcal{G}_{\mu\nu} - 4 \frac{\mathcal{H}}{N} \mathcal{K}_{\mu\nu} \\ &\quad + \left( \frac{4}{\ell_\Lambda^2} + 4 \frac{\mathcal{H}^2}{N^2} + \frac{\partial_{\tilde{y}} \mathcal{H}}{N} \right) \mathcal{G}_{\mu\nu} - a^{-2} R_{\mu\nu} + a^{-2} \frac{N_{;\mu\nu}}{N} , \end{aligned} \quad (3.14)$$

$$\mathcal{K}_{\mu\nu} = -\frac{1}{2} \partial_{\tilde{y}} \mathcal{G}_{\mu\nu} , \quad (3.15)$$

$$\mathcal{K}^2 - 6 \frac{\mathcal{H}}{N} \mathcal{K} - \mathcal{K}_\nu^\mu \mathcal{K}_\mu^\nu = R - 12 \left( \frac{\mathcal{H}^2}{N^2} + \frac{1}{\ell_\Lambda^2} \right) . \quad (3.16)$$

We expand  $\mathcal{G}_{\mu\nu}^{(\pm)}$  as

$$\begin{aligned} \mathcal{G}_{\mu\nu}^{(\pm)} &= \bar{\mathcal{G}}_{\mu\nu} + \overline{\partial_{\tilde{y}} \mathcal{G}_{\mu\nu}} \tilde{y}^\pm + \frac{1}{2} \overline{\partial_{\tilde{y}}^2 \mathcal{G}_{\mu\nu}} (\tilde{y}^\pm)^2 + \dots \\ &= \bar{\mathcal{G}}_{\mu\nu} - 2\bar{\mathcal{K}}_{\mu\nu} \tilde{y}^\pm - \overline{\partial_{\tilde{y}} \mathcal{K}_{\mu\nu}} (\tilde{y}^\pm)^2 + \dots , \end{aligned} \quad (3.17)$$

where the overbar denotes the value evaluated at  $y = 0$ .

In the following, we only keep the lowest order in the gradient expansion, *i.e.*, the leading order of

$$\epsilon := |Ky_0| , \quad (3.18)$$

and calculate the effective action up to the quadratic order of the metric perturbation. We want to investigate the parameter region with  $1 \mp 2r_c^{(\pm)} \mathcal{H}_\pm > 0$ , in which both radion and the massive gravitons are healthy [21]. Under the gradient expansion  $\epsilon \ll 1$ , namely,  $|K|$  becomes  $\mathcal{O}(1/r_c^{(\pm)})$  at most. We are also interested in the regime in which the mass squared of the lowest mode of massive gravitons is comparable to the 4-dimensional momentum squared. In Ref. [21], the mass squared of the lowest mode of massive gravitons is evaluated as

$$m^2 \simeq \frac{1}{2r_c^{(+)} y_0} \left( 1 + \frac{1}{\chi} \right) \sim \mathcal{O}(|K|^2/\epsilon) , \quad (3.19)$$

at the lowest order of the gradient expansion. Therefore,  $H^2$  and the 4-dimensional momentum squared become  $\mathcal{O}(|K|^2/\epsilon)$ , and then  $\ell_\Lambda^{-2}$  should be tuned so that  $\tilde{\ell}_\Lambda^{-2} = -\ell_\Lambda^{-2} + H^2 \sim \mathcal{O}(K^2)$  is much smaller than  $H^2 \sim \mathcal{O}(|K|^2/\epsilon)$ . This requires the tuning of the background expansion rate  $H$  to  $\approx \ell_\Lambda^{-1}$ , the value determined by the bulk cosmological constant. Furthermore, the Hamiltonian constraint (2.10) requires that  $R - 12\ell_\Lambda^{-2} \sim \mathcal{O}(K^2)$ . Also the junction conditions (2.6) will imply that the traceless part of  $R_{\mu\nu}$  becomes  $\mathcal{O}(K^2)$ , while the trace part of  $R_{\mu\nu}^{(\pm)}$  cancels with the brane tension  $\sigma_\pm$  at the leading order of the gradient expansion in Eq. (2.6). Therefore  $R_{\mu\nu}$  is given as

$$R_{\mu\nu} \sim 3\ell_\Lambda^{-2} g_{\mu\nu} + \mathcal{O}(K^2) . \quad (3.20)$$

From these conditions, the amplitudes of the matter energy momentum tensors on the respective branes are severely restricted, which does not allow us to use the present model to describe the system in which the background energy scale varies by a large amount.

With this understanding, we evaluate  $\mathcal{K}_{\mu\nu}$  at the middle point  $y = 0$  from Eq. (3.17) as

$$\begin{aligned} \bar{\mathcal{K}}_{\mu\nu} &= -\frac{\mathcal{G}_{\mu\nu}^{(+)} - \mathcal{G}_{\mu\nu}^{(-)}}{4\tilde{y}^+} + \dots \\ &= \left( \frac{x}{\tilde{\ell}_\Lambda} \gamma_{\mu\nu} + \frac{1}{4y_0} \Delta h_{\mu\nu} \right) (1 - x) + \mathcal{O}(\epsilon h K) , \end{aligned} \quad (3.21)$$

assuming that  $x \sim \mathcal{O}(\epsilon^0 h)$ , which will be confirmed later. Here we define

$$\Delta h_{\mu\nu} := h_{\mu\nu}^{(+)} - h_{\mu\nu}^{(-)} \sim \mathcal{O}(\epsilon h) . \quad (3.22)$$

The first term in the parentheses in Eq. (3.21) comes from the factor  $a^{-2}(y)$  in the definition of  $\mathcal{G}_{\mu\nu}$  (3.12). In order to raise the subscript of  $\bar{\mathcal{K}}_{\mu\nu}$  as well as to compute  $\bar{R}$ , we also need  $\bar{\mathcal{G}}_{\mu\nu}$  evaluated at the middle point, which is also computed from (3.17) as

$$\begin{aligned} \bar{\mathcal{G}}_{\mu\nu} &= \frac{1}{2} \left( \mathcal{G}_{\mu\nu}^{(+)} + \mathcal{G}_{\mu\nu}^{(-)} \right) + \overline{\partial_{\tilde{y}} \mathcal{K}_{\mu\nu}} (\tilde{y}^+)^2 + \dots \\ &= \gamma_{\mu\nu} + \tilde{h}_{\mu\nu} + 2H^2 x y_0^2 \gamma_{\mu\nu} + H^2 (x y_0)^2 \gamma_{\mu\nu} \\ &\quad + (1 + x) (\nabla_\mu \nabla_\nu x) y_0^2 + \mathcal{O}(\epsilon^2, \epsilon h) , \end{aligned} \quad (3.23)$$



where  $\nabla_\mu$  is the covariant differentiation associated with  $\gamma_{\mu\nu}$  and we define

$$\tilde{h}_{\mu\nu} := \frac{1}{2} \left( h_{\mu\nu}^{(+)} + h_{\mu\nu}^{(-)} \right). \quad (3.24)$$

Here we also use Eq. (3.14) but only its last three terms where  $\ell_\Lambda^{-2}$  and  $\bar{R}_{\mu\nu}$  are replaced by  $H^2$  and  $3H^2\bar{g}_{\mu\nu}$ , respectively, contribute to (3.23) at the order of the present approximation. At the 1st order of the metric perturbation  $h$ , the Hamiltonian constraint (3.16) determines  $x^{(1)}$  as

$$x^{(1)} = -\frac{\tilde{\ell}_\Lambda}{16y_0}\Delta h - \frac{\tilde{\ell}_\Lambda^2}{24}\bar{R}^{(1)} + \mathcal{O}(\epsilon h) =: -\frac{\tilde{\ell}_\Lambda}{16y_0}\Phi + \mathcal{O}(\epsilon h), \quad (3.25)$$

where  $x$  and  $\bar{R}$  are expanded as  $A = A^{(1)} + A^{(2)} + \dots$  with respect to order of the metric perturbation  $h$ . The last equality defines a new variable  $\Phi$ , which represents the rescaled perturbation of the brane separation.  $\bar{R}^{(1)}$  is obtained from Eq. (3.23) as

$$\begin{aligned} \bar{R}^{(1)} &= \mathcal{L}^{\mu\nu} \left( \tilde{h}_{\mu\nu} + 2H^2 y_0^2 x^{(1)} \gamma_{\mu\nu} \right) + \mathcal{O}(\epsilon h K^2) \\ &= \frac{1}{1 - \hat{H}^2 \hat{\square}} \mathcal{L}^{\mu\nu} \left( \tilde{h}_{\mu\nu} - \frac{\hat{H}^2}{4} \Delta h \gamma_{\mu\nu} \right) + \mathcal{O}(\epsilon h K^2), \end{aligned} \quad (3.26)$$

where we define

$$\mathcal{L}_{\mu\nu} := \nabla_\mu \nabla_\nu - \frac{\square}{4} \gamma_{\mu\nu} - \frac{3}{4} (\square + 4H^2) \gamma_{\mu\nu}, \quad \square := \nabla^\mu \nabla_\mu, \quad (3.27)$$

and

$$\alpha := \frac{y_0 \tilde{\ell}_\Lambda}{2}, \quad \hat{H}^2 := \alpha H^2, \quad \hat{\square} := \alpha (\square + 4H^2).$$

To arrive at the last expression in Eq. (3.26), we formally solved the differential equation obtained by substituting Eq. (3.25) into the first equality in Eq. (3.26), and we also use the identity

$$\mathcal{L}_{\mu\nu} \nabla^\mu A^\nu = 0, \quad (3.28)$$

which holds for an arbitrary vector  $A^\nu$ . Here, we need to discuss the boundary conditions for the inverse of the differential operator  $1 - \hat{H}^2 \hat{\square}$ . One may think that the non-local operator  $(1 - \hat{H}^2 \hat{\square})^{-1}$  may bring an extra degree of freedom corresponding to the pole at  $\hat{\square} = \hat{H}^{-2}$ . However, we know that there is no physical mode at  $\hat{\square} = \hat{H}^{-2}$  from the linear perturbation analysis [21, 33]. In order to avoid the appearance of such an unphysical degree of freedom, we should understand that the non-local operator  $(1 - \hat{H}^2 \hat{\square})^{-1}$  is to be expanded as  $1 + \hat{H}^2 \hat{\square} + \hat{H}^4 \hat{\square}^2 + \dots$  by restricting the energy scale to  $\hat{H}^2 \hat{\square} < 1$ . When  $H \approx m$  and the normal branch is chosen:  $1 \mp 2r_c^{(\pm)} \mathcal{H}_\pm > 0$ , the allowed energy region where our approximation is justified is estimated as

$$\square + 4H^2 \lesssim \mathcal{O} \left( K^2 \frac{r_c^{(+)}}{y_0} \right) \lesssim \mathcal{O}(m^2), \quad (3.29)$$

which is marginally consistent with the energy scale we wish to discuss. Interestingly, we will find later that, even if we keep this pole at  $\hat{\square} = \hat{H}^{-2}$ , its contribution to the metric

perturbation is not sourced by the matter energy momentum tensors localized on the branes. Substituting Eq. (3.26) into Eq. (3.25), we obtain

$$\Phi = -\frac{16y_0}{\tilde{\ell}_\Lambda} x^{(1)} = \Delta h + \frac{4\alpha}{3} \frac{1}{1 - \hat{H}^2 \hat{\square}} \mathcal{L}^{\mu\nu} \left( \tilde{h}_{\mu\nu} - \frac{\hat{H}^2}{4} \Delta h \gamma_{\mu\nu} \right). \quad (3.30)$$

Equipped with the solution to the linear order in  $\epsilon$ , we can evaluate the Hamiltonian constraint (3.16) at the 2nd order of  $h$  as

$$\begin{aligned} & -\frac{6}{\tilde{\ell}_\Lambda} \left[ \frac{4x^{(2)}}{\tilde{\ell}_\Lambda} - \left( \frac{\Delta h}{4y_0} + \frac{4x^{(1)}}{\tilde{\ell}_\Lambda} \right) x^{(1)} \right] \\ & - \frac{12x^{(1)2}}{\tilde{\ell}_\Lambda^2} - \frac{x^{(1)} \Delta h}{2\tilde{\ell}_\Lambda y_0} - \frac{\Delta h^{\mu\nu} \Delta h_{\mu\nu}}{16y_0^2} + \mathcal{O}(\epsilon h^2 K^2) = \bar{R}^{(2)}, \end{aligned} \quad (3.31)$$

where  $\bar{R}^{(2)}$  can be more explicitly expressed using Eq. (3.23) as

$$\bar{R}^{(2)} = y_0^2 \mathcal{L}^{\mu\nu} \left[ x^{(1)} \nabla_\mu \nabla_\nu x^{(1)} + H^2 \left( x^{(1)} \right)^2 \gamma_{\mu\nu} + 2H^2 x^{(2)} \gamma_{\mu\nu} \right] + \mathcal{O}(\epsilon h^2 K^2). \quad (3.32)$$

Then, as in the case of  $x^{(1)}$ ,  $x^{(2)}$  is solved as

$$\begin{aligned} x^{(2)} = \frac{1}{1 - \hat{H}^2 \hat{\square}} \frac{\tilde{\ell}_\Lambda^2}{384y_0^2} & \left[ \Delta h^2 - \Delta h^{\mu\nu} \Delta h_{\mu\nu} - \frac{3}{4} \Phi^2 + 4\alpha \Phi \bar{R}^{(1)} \right. \\ & \left. - \frac{\alpha^2}{4} \mathcal{L}^{\mu\nu} (\Phi \nabla_\mu \nabla_\nu \Phi + H^2 \Phi^2 \gamma_{\mu\nu}) \right] + \mathcal{O}(\epsilon h^2). \end{aligned} \quad (3.33)$$

Therefore, the total second-order effective action

$$\begin{aligned} S = \frac{M_{pl}^2}{2} & \left[ \frac{1}{2r_c^{(+)}} \oint d^5x \sqrt{-g} \left( R - \frac{12}{\ell_\Lambda^2} \right) \right. \\ & \left. + \int d^4x \sqrt{-g_{(+)}} \left( R_{(+)} - \frac{6H^2}{a_+^2} \right) + \chi \int d^4x \sqrt{-g_{(-)}} \left( R_{(-)} - \frac{6H^2}{a_-^2} \right) \right], \end{aligned} \quad (3.34)$$

is expressed in terms of  $g_{\mu\nu}^{(\pm)}$  as

$$\begin{aligned} & \frac{1}{2r_c^{(+)}} \oint d^5x \sqrt{-g} \left( R - \frac{12}{\ell_\Lambda^2} \right) \\ & = \frac{1}{r_c^{(+)}} \int_{y_0^+}^{-y_0^+} N dy \int d^4x \sqrt{-\gamma} \left\{ \frac{12}{\ell_\Lambda^2} + \mathcal{L}^{\mu\nu} (\bar{g}_{\mu\nu} - \tilde{y}^2 \partial_{\tilde{y}} \bar{K}_{\mu\nu}) + \mathcal{O}(\epsilon h^2 K^2) \right\} \\ & = \frac{1}{r_c^{(+)}} \int d^4x \sqrt{-\gamma} \left\{ 2Ny_0 \left( \frac{12}{\ell_\Lambda^2} + \bar{R}^{(1)} + \bar{R}^{(2)} \right) \right. \\ & \quad \left. - \frac{2}{3} Ny_0^3 \mathcal{L}^{\mu\nu} (N \nabla_\mu \nabla_\nu N + N^2 H^2 \gamma_{\mu\nu}) + \mathcal{O}(\epsilon^2 h^2 K) \right\} \\ & = \int d^4x \sqrt{-\gamma} \frac{1}{8r_c^{(+)} y_0} \left( 384 \left( \frac{y_0}{\ell_\Lambda} \right)^2 (1 - 4\hat{H}^4) x^{(2)} - 4\alpha \Phi \bar{R}^{(1)} + \mathcal{O}(\epsilon^3 h^2) \right) \\ & = \int d^4x \sqrt{-\gamma} m_*^2 \left\{ \Delta h^2 - \Delta h^{\mu\nu} \Delta h_{\mu\nu} - \frac{3}{4} \Phi (1 - \hat{H}^2 \hat{\square}) \Phi + \mathcal{O}(\epsilon^3 h^2) \right\}, \end{aligned} \quad (3.35)$$

where we define

$$m_*^2 := \frac{1}{8r_c^{(+)}y_0}. \quad (3.36)$$

In the first equality in Eq. (3.35),  $R(y) - 12\ell_\Lambda^{-2}$  is expanded around  $y = 0$  as

$$\begin{aligned} R(y) - \frac{12}{\ell_\Lambda^2} &= 12H^2 + \mathcal{L}^{\mu\nu} g_{\mu\nu}(y) - \frac{12}{\ell_\Lambda^2} + \mathcal{O}(\epsilon K^2) \\ &= \frac{12}{\ell_\Lambda^2} + \mathcal{L}^{\mu\nu} (\bar{g}_{\mu\nu} - 2\tilde{y}\bar{K}_{\mu\nu} - \tilde{y}^2\overline{\partial_{\tilde{y}}K_{\mu\nu}} + \dots) + \mathcal{O}(\epsilon K^2), \end{aligned} \quad (3.37)$$

where the linear term in  $\tilde{y}$  is integrated to be zero in the integral. In the third equality in Eq. (3.35), we use Eqs. (3.28) and (3.32) and neglect the total derivative terms. The action (3.34) contains  $(1 - \hat{H}^2\hat{\square})^{-1}$  through  $\Phi$  defined by Eq. (3.30). As mentioned above, we treat the apparently non-local operator  $(1 - \hat{H}^2\hat{\square})^{-1}$  as a local one by imposing the restriction presented in Eq. (3.29).

In the extreme limit of self-accelerating branch  $\alpha m_*^2 \rightarrow 0$ , which corresponds to the limit  $r_c^{(+)}\mathcal{H}_+ \rightarrow \infty$ ,  $\Phi$  is reduced to  $\Delta h$  and the bulk contribution in the effective action (3.35) becomes

$$\frac{M_{pl}^2 m_*^2}{2} \int d^4x \sqrt{-\gamma} \left( \frac{1}{4} \Delta h^2 - \Delta h_{\mu\nu} \Delta h^{\mu\nu} \right),$$

which differs from the form of the ghost-free dRGT mass terms. Therefore, the system possesses an extra scalar mode in addition to the two gravitons and a BD ghost appears. In the self-accelerating branch of DGP 2-brane model, we know that either the extra scalar mode corresponding to the radion or the helicity-0 mode of the massive graviton becomes ghost [32, 33]. We refer to them as the radion ghost and the Higuchi ghost, respectively. The appearance of a BD ghost here seems to be exactly corresponding to the inevitable existence of either the radion ghost or the Higuchi ghost. When the mass squared of the massive graviton is smaller than  $2H^2$ , the ghost in the DGP model is the Higuchi ghost, and the parameter region with such a small graviton mass is not excluded. Hence, this example shows that the BD ghost can be the Higuchi ghost. The existence of the Higuchi ghost may not be so harmful as a usual scalar ghost, as discussed in Ref. [35].

Now we return to the case without taking the limit  $\alpha m_*^2 \rightarrow 0$ , and show in general that the system described by the total effective action (3.34) consists of one massless graviton, one massive graviton, and one scalar. We rewrite the action (3.34) so that the rescaled perturbation of the brane separation,  $\Phi$ , is manifestly treated as an independent degree of freedom, radion. For this purpose, we introduce a Lagrange multiplier  $\lambda$  to impose the constraint  $\Phi = \Delta h + \frac{4\alpha}{3}\bar{R}^{(1)}$  as

$$\begin{aligned} S = \frac{M_{pl}^2}{2} \left[ \int d^4x \sqrt{-\gamma} m_*^2 \left\{ \Delta h^2 - \Delta h^{\mu\nu} \Delta h_{\mu\nu} - \frac{3}{4} \Phi (1 - \hat{H}^2\hat{\square}) \Phi + \lambda \left( \Phi - \Delta h - \frac{4\alpha}{3} \bar{R}^{(1)} \right) \right\} \right. \\ \left. + \int d^4x \sqrt{-g_{(+)}} \left( R_{(+)} - \frac{6H^2}{a_+^2} \right) + \chi \int d^4x \sqrt{-g_{(-)}} \left( R_{(-)} - \frac{6H^2}{a_-^2} \right) \right]. \end{aligned} \quad (3.38)$$

Taking the variation with respect to  $\Phi$ , we obtain a simple expression for  $\lambda$  written in terms of  $\Phi$ :

$$\lambda = \frac{3}{2} (1 - \hat{H}^2\hat{\square}) \Phi. \quad (3.39)$$

Then, substituting this  $\lambda$  back into the action (3.38), the action written in terms of  $h_{\mu\nu}^{(\pm)}$  and  $\Phi$  is obtained as

$$S = \frac{M_{pl}^2}{2} \left[ \int d^4x \sqrt{-\gamma} m_*^2 \left\{ \Delta h^2 - \Delta h^{\mu\nu} \Delta h_{\mu\nu} - \frac{3}{4} \alpha^2 H^2 \Phi (\square + 4H^2) \Phi \right. \right. \\ \left. \left. + \frac{3}{4} \Phi (\Phi - 2\Delta h) - 2\alpha \Phi \mathcal{L}^{\mu\nu} \tilde{h}_{\mu\nu} \right\} \right. \\ \left. + \int d^4x \sqrt{-g_{(+)}} \left( R_{(+)} - \frac{6H^2}{a_+^2} \right) + \chi \int d^4x \sqrt{-g_{(-)}} \left( R_{(-)} - \frac{6H^2}{a_-^2} \right) \right]. \quad (3.40)$$

In this step,  $1 - \hat{H}^2 \hat{\square}$  in Eq. (3.39) cancels out the apparently non-local operator  $(1 - \hat{H}^2 \hat{\square})^{-1}$  in  $\bar{R}^{(1)}$ . After rewriting the effective action into this form, we can obtain the ghost-free Fierz-Pauli model even in the limit  $\alpha m_*^2 \rightarrow 0$ , by setting the radion  $\Phi$  to 0, assuming some steep stabilization potential for  $\Phi$  by hand. The last term in the second line of Eq. (3.40) can be absorbed into the induced gravity terms by the conformal transformations  $g_{\mu\nu}^{(+)} = \exp(m_*^2 \alpha \Phi) \hat{g}_{\mu\nu}^{(+)}$  and  $g_{\mu\nu}^{(-)} = \exp([m_*^2 \alpha / \chi] \Phi) \hat{g}_{\mu\nu}^{(-)}$ . Then, the action becomes

$$S = \frac{M_{pl}^2}{2} \left[ \int d^4x \sqrt{-\gamma} m_*^2 \left\{ \Delta h^2 - \Delta h^{\mu\nu} \Delta h_{\mu\nu} + \frac{3}{4} \Phi (\Phi - 2\Delta h) \right. \right. \\ \left. \left. + \frac{3}{2} \alpha^2 \left( m_*^2 \left( 1 + \frac{1}{\chi} \right) - \frac{1}{2} H^2 \right) \Phi (\square + 4H^2) \Phi \right\} \right. \\ \left. + \int d^4x \sqrt{-\hat{g}_{(+)}} \left( \hat{R}_{(+)} - \frac{6H^2}{a_+^2} \right) + \chi \int d^4x \sqrt{-\hat{g}_{(-)}} \left( \hat{R}_{(-)} - \frac{6H^2}{a_-^2} \right) \right], \quad (3.41)$$

where  $\hat{R}_{(+)}$  and  $\hat{R}_{(-)}$  are the Ricci scalars for the metrics  $\hat{g}_{\mu\nu}^{(+)}$  and  $\hat{g}_{\mu\nu}^{(-)}$ , respectively. Here, two gravitons interact with each other through the Fierz-Pauli mass term and the radion field  $\Phi$ , which is now an independent field, couples to both metrics, but its kinetic term couples to the metrics only through the combination  $\gamma_{\mu\nu}$ . To be honest, at the current level of our approximation in which all the terms higher order in  $\epsilon$  are neglected, we cannot discriminate to which metric the radion field is coupled. Therefore the model described by this effective action satisfies the BD-ghost-free conditions for doubly coupled fields in bigravity [25, 26]. Since the action (3.41) does not suffer from higher derivatives nor BD ghost problem, this system contains only two gravitons and one scalar radion. Because of the presence of the constraints for  $\Delta h_{\mu\nu}$ , it cannot be immediately confirmed by the action (3.41) whether the helicity-0 mode of the massive graviton and the scalar radion are healthy or not, which will be discussed in the next section by writing down the effective action in terms of the matter energy-momentum tensor, integrating out the gravitational degrees of freedom.

#### 4 Equations of motion and coupling to the matter energy momentum tensors on the branes

Now we analyze the equations of motion derived from the action (3.34) with additional matter fields localized on the branes. We confirm that the system contains only two gravitons and one scalar mode whose ghost-free condition is equivalent to the one discussed in Ref. [21], by investigating the poles of the propagators and their coefficients. Taking the variations of the

action (3.34) with respect to  $h_{\mu\nu}^{(\pm)}$ , we obtain the equations of motion:

$$\begin{aligned} \chi_{\pm} \mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta}^{(\pm)} \pm 2m_*^2 (\Delta h_{\mu\nu} - \Delta h \gamma_{\mu\nu}) \\ + \frac{3}{2} m_*^2 \left[ \pm \gamma_{\mu\nu} + \frac{2}{3} \alpha \mathcal{L}_{\mu\nu} \right] \left( \Delta h + \frac{4}{3} \alpha \bar{R}^{(1)} \right) = M_{pl}^{-2} T_{\mu\nu}^{(\pm)}, \end{aligned} \quad (4.1)$$

where we define  $\chi_+ = 1$ ,  $\chi_- = \chi$ , and

$$\begin{aligned} \mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} := -\frac{1}{2} \left( \square h_{\mu\nu} + \nabla_\mu \nabla_\nu h - 2 \nabla_{(\nu} \nabla_\sigma h_{\mu)}^\sigma - \gamma_{\mu\nu} \square h + \gamma_{\mu\nu} \nabla_\alpha \nabla_\beta h^{\alpha\beta} \right. \\ \left. - 2H^2 h_{\mu\nu} - H^2 \gamma_{\mu\nu} h \right), \end{aligned} \quad (4.2)$$

is the linearized Einstein equations derived from the variation of the four-dimensional localized Einstein-Hilbert part of the action. In the above expression, we set  $a_{\pm} = 1$ . In the present order of approximation, it is no use to distinguish  $a_{\pm}$  from unity. Here, we impose the gauge conditions

$$\nabla^\mu \left( \tilde{h}_{\mu\nu} - \frac{1}{4} \gamma_{\mu\nu} \tilde{h} \right) = 0, \quad (4.3)$$

on the averaged part of the metric perturbation (3.24), so that the traceless part of  $\tilde{h}_{\mu\nu}$  becomes transverse. On the other hand, we decompose the difference of the metric perturbations (3.22) as

$$\Delta h_{\mu\nu} = \Delta h_{\mu\nu}^{(TT)} + \frac{1}{4} \phi \gamma_{\mu\nu} + \left( \nabla_\mu \nabla_\nu - \frac{\square}{4} \gamma_{\mu\nu} \right) \psi, \quad (4.4)$$

where  $\Delta h_{\mu\nu}^{(TT)}$  is the transverse-traceless part of  $\Delta h_{\mu\nu}$ . Using the identities, (3.28) and

$$\nabla^\mu \mathcal{E}_{\mu\nu}^{\alpha\beta} H_{\alpha\beta} = 0, \quad (4.5)$$

$$\nabla^\mu \left( \nabla_\mu \nabla_\nu - \frac{\square}{4} \gamma_{\mu\nu} \right) \Psi = \frac{3}{4} \nabla_\nu (\square + 4H^2) \Psi, \quad (4.6)$$

$$\nabla^\mu \mathcal{L}_{\mu\nu} \Psi = 0, \quad (4.7)$$

where  $\Psi$  and  $H_{\alpha\beta}$  are arbitrary scalar and tensor, respectively, one can take the divergence of Eq. (4.1). Then, the two equations derived from the divergence of Eq. (4.1) give the same equation, from which  $\psi$  is determined as

$$\psi = \frac{\alpha}{2} Z, \quad (4.8)$$

where

$$Z := \left( 1 - \hat{H}^2 \hat{\square} \right)^{-1} \left( 2\tilde{h} - 2\hat{H}^2 \phi \right). \quad (4.9)$$

Here, we ignore the homogeneous solution that satisfies  $(\square + 4H^2) \psi = 0$  because it degenerates with the transverse-traceless mode  $\Delta h_{\mu\nu}^{(TT)}$  by the identity (4.6). One may think that  $\psi$  is not uniquely determined because of the pole at  $\hat{\square} = \hat{H}^{-2}$ . However, such a possible ambiguity has been already eliminated by imposing the restriction (3.29). By substituting Eq. (4.8), Eq. (4.1) gives

$$\chi_{\pm} \mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta}^{(\pm)} \pm 2m_*^2 \Delta h_{\mu\nu}^{(TT)} + m_*^2 \alpha \mathcal{L}_{\mu\nu} \left[ \phi \pm \left( 1 \mp \frac{\hat{\square}}{2} \right) Z \right] = M_{pl}^{-2} T_{\mu\nu}^{(\pm)}. \quad (4.10)$$

On the other hand, the trace of the metric perturbation on each brane  $h^{(\pm)}$  after eliminating the terms proportional to  $\nabla_\mu \nabla_\nu \psi$ , which can be erased by a 4-dimensional gauge transformation from  $h_{\mu\nu}^{(\pm)}$ , are found to be given by

$$\begin{aligned} 2h^{(\pm)} &:= 2\tilde{h} \pm (\phi - \square\psi) \\ &= \left(2\hat{H}^2 \pm 1\right) \left[ \phi \pm \left(1 \mp \frac{\hat{\square}}{2}\right) Z \right], \end{aligned} \quad (4.11)$$

where in the second equality we use Eq. (4.9) again. Using this equality and Eq. (3.8), the equations of motion for  $h_{\mu\nu}^{(\pm)}$  (4.10) are simplified as

$$\chi_\pm \mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta}^{(\pm)} \pm 2m_*^2 \Delta h_{\mu\nu}^{(TT)} + \left(\pm 8r_c^{(+)} \mathcal{H}_\pm\right)^{-1} \mathcal{L}_{\mu\nu} h^{(\pm)} = M_{pl}^{-2} T_{\mu\nu}^{(\pm)}. \quad (4.12)$$

The trace part of Eq. (4.12) becomes

$$h^{(\pm)} = \frac{4M_{pl}^{-2}}{3} \left( \chi_\pm \mp \frac{1}{2r_c^{(+)} \mathcal{H}_\pm} \right)^{-1} \frac{1}{\square + 4H^2} T^{(\pm)}. \quad (4.13)$$

Diagonalizing the traceless part of Eq. (4.12) with respect to the mass eigenvalues, we obtain the equations of motion for the transverse-traceless part of the massless mode  $h_{\mu\nu}^{(0)}$ ,

$$h_{\mu\nu}^{(0)} := (1 + \chi)^{-1} \left( h_{\mu\nu}^{(+)} + \chi h_{\mu\nu}^{(-)} \right), \quad (4.14)$$

and that of the massive mode  $h_{\mu\nu}^{(m)}$

$$h_{\mu\nu}^{(m)} := (1 + \chi)^{-1} \Delta h_{\mu\nu}, \quad (4.15)$$

as

$$\begin{aligned} (\square - 2H^2) h_{\mu\nu}^{(0)TT} &= \frac{-2M_{pl}^{-2}}{1 + \chi} \left[ T_{\mu\nu}^{(0)} - \frac{1}{4} T^{(0)} \gamma_{\mu\nu} \right. \\ &\quad \left. + \frac{1}{3} \left( \nabla_\mu \nabla_\nu - \frac{\square}{4} \gamma_{\mu\nu} \right) \frac{1}{\square + 4H^2} T^{(0)} \right], \end{aligned} \quad (4.16)$$

$$\begin{aligned} (\square - 2H^2 - m^2) h_{\mu\nu}^{(m)TT} &= \frac{-2M_{pl}^{-2}}{1 + \chi} \left[ T_{\mu\nu}^{(m)} - \frac{1}{4} T^{(m)} \gamma_{\mu\nu} \right. \\ &\quad \left. + \frac{1}{3} \left( \nabla_\mu \nabla_\nu - \frac{\square}{4} \gamma_{\mu\nu} \right) \frac{1}{\square + 4H^2} T^{(m)} \right], \end{aligned} \quad (4.17)$$

where  $m^2$  is defined in Eq. (3.19) and we define

$$T_{\mu\nu}^{(0)} := T_{\mu\nu}^{(+)} + T_{\mu\nu}^{(-)}, \quad (4.18)$$

$$T_{\mu\nu}^{(m)} := T_{\mu\nu}^{(+)} - \frac{T_{\mu\nu}^{(-)}}{\chi}, \quad (4.19)$$

Using the identity

$$\left( \nabla_\mu \nabla_\nu - \frac{\square}{4} \gamma_{\mu\nu} \right) \frac{1}{\square + 4H^2} \Psi = \frac{1}{\square - 4H^2} \left( \nabla_\mu \nabla_\nu - \frac{\square}{4} \gamma_{\mu\nu} \right) \Psi, \quad (4.20)$$

which holds for an arbitrary scalar  $\Psi$ ,  $h_{\mu\nu}^{(0)}$  and  $h_{\mu\nu}^{(m)}$  are found to be rewritten as

$$h_{\mu\nu}^{(0)TT} = \frac{-2M_{pl}^{-2}}{1+\chi} \left[ \frac{1}{\square - 2H^2} \left\{ T_{\mu\nu}^{(0)} - \frac{1}{4}T^{(0)}\gamma_{\mu\nu} + \frac{1}{3(-2H^2)} \left( \nabla_\mu \nabla_\nu - \frac{\square}{4}\gamma_{\mu\nu} \right) T^{(0)} \right\} \right. \\ \left. - \frac{1}{3(-2H^2)} \left( -\frac{\square}{4}\gamma_{\mu\nu} \right) \frac{1}{\square + 4H^2} T^{(0)} \right], \quad (4.21)$$

$$h_{\mu\nu}^{(m)TT} = \frac{-2M_{pl}^{-2}}{1+\chi} \left[ \frac{1}{\square - 2H^2 - m^2} \left\{ T_{\mu\nu}^{(m)} - \frac{1}{4}T^{(m)}\gamma_{\mu\nu} \right. \right. \\ \left. \left. + \frac{1}{3(m^2 - 2H^2)} \left( \nabla_\mu \nabla_\nu - \frac{\square}{4}\gamma_{\mu\nu} \right) T^{(m)} \right\} \right. \\ \left. - \frac{1}{3(m^2 - 2H^2)} \left( -\frac{\square}{4}\gamma_{\mu\nu} \right) \frac{1}{\square + 4H^2} T^{(m)} \right], \quad (4.22)$$

where we eliminate some terms which can be erased by a 4-dim gauge transformation as in Eq. (4.11). If we set  $T_{\mu\nu}^{(-)} = 0$ , for simplicity, the coefficient of the pole  $(\square + 4H^2)^{-1}$  in  $h_{\mu\nu}^{(+)} = h_{\mu\nu}^{(0)} + \chi h_{\mu\nu}^{(m)}$  is extracted as

$$h_{\mu\nu}^{(+)} \supset \frac{M_{pl}^{-2}}{3} \left[ \frac{2r_c^{(+)}\mathcal{H}_+}{2r_c^{(+)}\mathcal{H}_+ - 1} + \frac{1}{1+\chi} \left( -1 + \frac{2\chi H^2}{m^2 - 2H^2} \right) \right] \frac{1}{\square + 4H^2} T^{(+)} \gamma_{\mu\nu}. \quad (4.23)$$

When we integrate out the scalar degree of freedom, the contribution of the pole at  $\square = -4H^2$  to the effective action is written in terms of  $T_{\mu\nu}^{(+)}$  as

$$- \int d^4x \sqrt{-\gamma} h_{\mu\nu}^{(+)} T_{(+)}^{\mu\nu} \supset \frac{M_{pl}^{-2}}{3} \beta_+ \int d^4x \sqrt{-\gamma} T^{(+)} \frac{1}{\square + 4H^2} T^{(+)}, \quad (4.24)$$

where

$$\beta_+ := \frac{1}{1 - 2r_c^{(+)}\mathcal{H}_+} - \frac{\chi}{1+\chi} \frac{m^2}{m^2 - 2H^2}, \quad (4.25)$$

is the minus of the part embraced by the square brackets in Eq. (4.23). The sign of  $\beta_+$  determines whether the scalar degree of freedom is ghost or not. When  $\beta_+$  is negative, the scalar degree of freedom is ghost. When the condition  $1 - 2r_c^{(+)}\mathcal{H}_+ > 0$ , which is one of the two ghost-free branch conditions derived in Ref. [21], is satisfied, the first term in the square brackets in Eq. (4.25) is positive and greater than unity. If  $\mathcal{H}_+$  is positive, for sufficiently large  $m^2$ , the absolute value of the second term becomes less than unity, and hence  $\beta_+$  becomes positive in total. When  $1 - 2r_c^{(+)}\mathcal{H}_+ < 0$ , the positiveness of  $\beta_+$  imposes  $m^2 - 2H^2 < 0$ , namely, Higuchi ghost appears.

A similar discussion applies to the case where  $T_{\mu\nu}^{(+)} = 0$ . The coefficient of the pole  $(\square + 4H^2)^{-1}$  in  $h_{\mu\nu}^{(-)}$  becomes

$$- \int d^4x \sqrt{-\gamma} h_{\mu\nu}^{(-)} T_{(-)}^{\mu\nu} \supset \frac{M_{pl}^{-2}}{3} \beta_- \int d^4x \sqrt{-\gamma} T^{(-)} \frac{1}{\square + 4H^2} T^{(-)}, \quad (4.26)$$

where

$$\beta_- := \frac{1}{\chi} \left[ \frac{1}{1 + 2r_c^{(-)}\mathcal{H}_-} - \frac{1}{1+\chi} \frac{m^2}{m^2 - 2H^2} \right]. \quad (4.27)$$

Thus, the contribution of the pole at  $\square = -4H^2$  to the effective action is positive and free from Higuchi ghost when  $1 + 2r_c^{(-)}\mathcal{H}_- > 0$  and the first term is larger than the second term in magnitude. The first condition is identical to the other ghost-free-branch condition derived in Ref. [21], as is expected.

Suppose that for sufficiently large  $m^2$ , the ghost-free conditions are satisfied. If we set  $m^2$  is larger than and close to  $2H^2$ , both  $\beta_+$  and  $\beta_-$  necessarily become negative. Hence, there should be critical values of  $m^2$  at which  $\beta_+$  or  $\beta_-$  cross zero. Since the scalar mode couples to both traces of the energy momentum tensors,  $T_{(+)}$  and  $T_{(-)}$ , we expect that the signatures of  $\beta_+$  and  $\beta_-$  should flip simultaneously. In fact, at the leading order of the gradient expansion, we can rewrite  $\beta_{\pm}$  using Eqs. (3.8) and (3.19) as

$$\beta_{\pm} = \frac{1}{1 + \chi} \frac{m^2}{m^2 - 2H^2} \frac{1 \pm 2r_c^{(\mp)}\mathcal{H}_{\mp}}{1 \mp 2r_c^{(\pm)}\mathcal{H}_{\pm}}. \quad (4.28)$$

Then, we find that the condition  $\beta_{\pm} > 0$  is equivalent to  $1 \pm 2r_c^{(\mp)}\mathcal{H}_{\mp} > 0$ , when  $m^2 > 2H^2$  and  $1 \mp 2r_c^{(\pm)}\mathcal{H}_{\pm} > 0$ . This simultaneous sign flip occurs at the transition from the ghost-free normal branch to the self-accelerating branch. At the transition point in the parameter space, either  $\beta_+$  or  $\beta_-$  diverges. (If we assume  $\mathcal{H}_+ \approx \mathcal{H}_- > 0$ ,  $\beta_-$  never diverges. Then, the divergent is necessarily  $\beta_+$ .) Therefore, the transition between two branches is accompanied by the divergence of metric perturbation and the perturbative approach breaks down at the branch crossing point.

Let us examine how one can construct the background spacetime that satisfies the conditions for the normal branch. Using the normal-branch conditions  $1 \mp 2r_c^{(\pm)}\mathcal{H}_{\pm} \geq 0$  and Eqs. (3.8) and (3.19), we obtain a constraint for  $\tilde{\ell}_{\Lambda}$  as

$$-1 + (1 + \chi) \frac{H^2}{m^2} < \frac{2r_c^{(-)}}{\tilde{\ell}_{\Lambda}} < \chi - (1 + \chi) \frac{H^2}{m^2}. \quad (4.29)$$

In order to satisfy this constraint,  $-1 + (1 + \chi) H^2/m^2 < \chi - (1 + \chi) H^2/m^2$  must be satisfied. Thus, the mass squared of the massive graviton (3.19), which can be adjusted by changing  $y_0^+$ , must be tuned to be above the Higuchi bound, *i.e.*,  $m^2 > 2H^2$ . When  $m^2 > 2H^2$ , Eq. (4.29) can be achieved by tuning  $\tilde{\ell}_{\Lambda}$  depending on  $H/m$  and  $r_c^{(\pm)}$ .

The physical meaning of the normal branch conditions can be made clear in the following way. Eliminating  $H^2/a_{\pm}^2$  from Eq. (3.3) with (3.2), we obtain the relations among  $\mathcal{H}_{\pm}$ ,  $\sigma_{\pm}$  and  $\ell_{\Lambda}$  as

$$\mathcal{H}_{\pm}^2 \mp \frac{\mathcal{H}_{\pm}}{r_c^{(\pm)}} + \frac{1}{\ell_{\Lambda}^2} - \frac{\sigma_{\pm}}{3} = 0. \quad (4.30)$$

Then, we find that there are two solutions for  $\mathcal{H}_{\pm}$  as

$$1 - 2r_c^{(+)}\mathcal{H}_+ = \pm \sqrt{1 - 4r_c^{(+)^2} \left( \frac{1}{\ell_{\Lambda}^2} - \frac{\sigma_+}{3} \right)}, \quad (4.31)$$

$$1 + 2r_c^{(-)}\mathcal{H}_- = \pm \sqrt{1 - 4r_c^{(-)^2} \left( \frac{1}{\ell_{\Lambda}^2} - \frac{\sigma_-}{3} \right)}, \quad (4.32)$$

when  $(r_c^{(\pm)})^{-2} - 4(\ell_{\Lambda}^{-2} - \sigma_{\pm}/3) > 0$ . As is discussed above, the solution that takes the positive sign on both branes corresponds to the normal branch. If either of the above solutions takes



negative sign, the whole setup is in the self-accelerating branch. If we have chosen that  $\mathcal{H}$  is always positive, the sign in Eq. (4.32) is guaranteed to be positive by assumption. The above two solutions (4.31) degenerate to one double root when  $(r_c^{(\pm)})^{-2} - 4(\ell_\Lambda^{-2} - \sigma_\pm/3) = 0$ , and consequently we have  $1 \mp 2r_c^{(\pm)}\mathcal{H}_\pm = 0$  in this case. For  $1 - 2r_c^{(+)}\mathcal{H}_+ = 0$  ( $1 + 2r_c^{(-)}\mathcal{H}_- = 0$ ), the radion strongly couples to the source on the (+)-brane ((-)-brane) as is seen in Eq. (4.25) (Eq. (4.27)). This can be understood as a consequence of the perplexity about the branch choice by the system.

## 5 Nonlinear generalization

We extend the above method, with which we succeeded in reproducing a healthy action for bigravity with a scalar corresponding to the radion as a low-energy effective theory concerning the perturbation around a de Sitter spacetime, to the nonlinear perturbation, although the gradient expansion is not extended to the higher order. Here we assume that

$$\epsilon := |K\tilde{y}^+|, \quad (5.1)$$

is small as before with  $g_{\mu\nu}^{(\pm)} \sim \mathcal{O}(1)$ ,  $\Delta g_{\mu\nu} := g_{\mu\nu}^{(+)} - g_{\mu\nu}^{(-)} \sim \mathcal{O}(\epsilon)$ , and  $\nabla^2 \sim \ell_\Lambda^{-2} \sim \mathcal{O}(|K|^2/\epsilon)$ . Furthermore, the Hamiltonian constraint (2.10) and the junction conditions (2.6) imply Eq. (3.20), as is discussed in Sec. 3. As in Sec. 3, we obtain  $\bar{K}_{\mu\nu}$  and  $\bar{g}_{\mu\nu}$  from Eq. (2.13) as

$$\bar{K}_{\mu\nu} = \frac{\Delta g_{\mu\nu}}{4\Phi}, \quad (5.2)$$

$$\bar{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + \Phi \nabla_\mu \nabla_\nu \Phi + \frac{\Phi^2}{\ell_\Lambda^2} g_{\mu\nu} + \mathcal{O}(\epsilon^2), \quad (5.3)$$

where

$$\Phi := Ny_0, \quad (5.4)$$

$$\tilde{g}_{\mu\nu} := \frac{g_{\mu\nu}^{(+)} + g_{\mu\nu}^{(-)}}{2}. \quad (5.5)$$

The Hamiltonian constraint (2.10) becomes

$$\bar{R} - \frac{12}{\ell_\Lambda^2} = \frac{\Delta g^2 - \Delta g_{\mu\nu} \Delta g^{\mu\nu}}{16\Phi^2}, \quad (5.6)$$

where the tensor indices are raised by  $\tilde{g}_{\mu\nu}$ . The bulk action is given as

$$\begin{aligned}
S_b &= \frac{M_{pl}^2}{2} \frac{1}{2r_c^{(+)}} \oint \sqrt{-g} d^4x N dy \left( {}^5R - \frac{12}{\ell_\Lambda^2} \right) \\
&= \frac{M_{pl}^2}{2} \frac{1}{r_c^{(+)}} \int d^4x N \int_{-y_0}^{y_0} dy \left[ \sqrt{-\bar{g}} \left( \bar{R} - \frac{12}{\ell_\Lambda^2} \right) + y \frac{\overline{\delta \left( \sqrt{-g} \left( R - \frac{12}{\ell_\Lambda^2} \right) \right)}}{\delta g_{\mu\nu}} \overline{\partial_y g_{\mu\nu}} \right. \\
&\quad \left. + \frac{y^2}{2} \partial_y \left( \frac{\overline{\delta \left( \sqrt{-g} \left( R - \frac{12}{\ell_\Lambda^2} \right) \right)}}{\delta g_{\mu\nu}} \partial_y g_{\mu\nu} \right) + \dots \right] \\
&= \frac{M_{pl}^2}{2} \frac{1}{r_c^{(+)}} \int d^4x N \left[ 2y_0 \sqrt{-\bar{g}} \left( \bar{R} - \frac{12}{\ell_\Lambda^2} \right) \right. \\
&\quad \left. - \frac{2y_0^3}{3} \frac{\overline{\delta \left( \sqrt{-g} \left( R - \frac{12}{\ell_\Lambda^2} \right) \right)}}{\delta g_{\mu\nu}} N^2 \left( \frac{1}{\Phi} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Phi + \frac{1}{\ell_\Lambda^2} \tilde{g}_{\mu\nu} \right) \right. \\
&\quad \left. + \mathcal{O}(|K|\epsilon^2) \right], \tag{5.7}
\end{aligned}$$

where we use the Hamiltonian constraint (2.10) in the second equality. In the third equality, the linear term in  $y$  in the integral is integrated to be zero, the term in which differentiation with respect to  $y$  operates on  $\delta \left( \sqrt{-g} \left( R - 12\ell_\Lambda^{-2} \right) \right) / \delta g_{\mu\nu}$  becomes higher order of the gradient expansion, and we use Eq. (2.8) and  $\tilde{R}_{\mu\nu} = 3\ell_\Lambda^{-2} \tilde{g}_{\mu\nu} + \mathcal{O}(|K|^2)$ . The operator  $\overline{\delta \left( \sqrt{-g} \left( R - 12\ell_\Lambda^{-2} \right) \right) / \delta g_{\mu\nu}}$  is obtained as

$$\int d^4x \frac{\overline{\delta \left( \sqrt{-g} \left( R - \frac{12}{\ell_\Lambda^2} \right) \right)}}{\delta g_{\mu\nu}} h_{\mu\nu} = \int d^4x \left( \sqrt{-\tilde{g}} \tilde{\mathcal{L}}^{\mu\nu} h_{\mu\nu} + \mathcal{O}(|K|^2) \right), \tag{5.8}$$

for an arbitrary tensor  $h_{\mu\nu}$ . Here we defined  $\tilde{\mathcal{L}}^{\mu\nu}$  as

$$\tilde{\mathcal{L}}^{\mu\nu} := \tilde{\nabla}^\mu \tilde{\nabla}^\nu - \left( \tilde{\square} + \frac{3}{\ell_\Lambda^2} \right) \tilde{g}^{\mu\nu}, \tag{5.9}$$

where  $\tilde{\nabla}$  is the covariant differentiation associated with  $\tilde{g}_{\mu\nu}$ . Using the Hamiltonian constraint Eq. (5.6), the bulk action becomes

$$\begin{aligned}
S_b &= \frac{M_{pl}^2}{2} \frac{2}{r_c^{(+)}} \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\Delta g^2 - \Delta g_{\mu\nu} \Delta g^{\mu\nu}}{16\Phi} + \frac{\Phi}{3} \tilde{\mathcal{L}}^{\mu\nu} \left( \left( \tilde{\nabla}_\mu \Phi \right) \left( \tilde{\nabla}_\nu \Phi \right) - \frac{\Phi^2}{\ell_\Lambda^2} \tilde{g}_{\mu\nu} \right) \right. \\
&\quad \left. + \mathcal{O}(|K|\epsilon^2) \right], \tag{5.10}
\end{aligned}$$

where we use  $\tilde{\mathcal{L}}^{\mu\nu} \tilde{\nabla}_\mu A_\nu = 0$  for an arbitrary vector  $A_\nu$  at the leading order of the expansion in  $\epsilon$ . The analysis in the linear regime suggests that in order to examine the structure of the interaction between  $g_{\mu\nu}^{(+)}$  and  $g_{\mu\nu}^{(-)}$ , it is convenient to treat the radion  $\Phi$  as an independent variable. Therefore we consider

$$\begin{aligned}
S_b &= \frac{M_{pl}^2}{2} \frac{2}{r_c^{(+)}} \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\Delta g^2 - \Delta g_{\mu\nu} \Delta g^{\mu\nu}}{16\Phi} + \frac{\Phi}{3} \tilde{\mathcal{L}}^{\mu\nu} \left( \left( \tilde{\nabla}_\mu \Phi \right) \left( \tilde{\nabla}_\nu \Phi \right) - \frac{\Phi^2}{\ell_\Lambda^2} \tilde{g}_{\mu\nu} \right) \right. \\
&\quad \left. + \lambda \left( \bar{R} - \frac{12}{\ell_\Lambda^2} - \frac{\Delta g^2 - \Delta g_{\mu\nu} \Delta g^{\mu\nu}}{16\Phi^2} \right) \right], \tag{5.11}
\end{aligned}$$

by imposing the Hamiltonian constraint with a Lagrange multiplier  $\lambda$ , and take the variation with respect to  $\Phi$  to eliminate  $\lambda$  using the equation of motion for  $\Phi$ .  $\bar{R} - 12\ell_\Lambda^{-2}$  is computed as

$$\bar{R} - \frac{12}{\ell_\Lambda^2} = \tilde{R} - \frac{12}{\ell_\Lambda^2} + \tilde{\mathcal{L}}^{\mu\nu} \left( \Phi \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Phi + \frac{\Phi^2}{\ell_\Lambda^2} \tilde{g}_{\mu\nu} \right) + \mathcal{O}(|K|\epsilon). \quad (5.12)$$

By taking the variation of Eq. (5.11) with respect to  $\Phi$  and using the identity  $\tilde{\mathcal{L}}_{\mu\nu} \tilde{\nabla}^\mu \tilde{\nabla}^\nu \Psi = \tilde{\nabla}^\mu \tilde{\nabla}^\nu \tilde{\mathcal{L}}_{\mu\nu} \Psi = 0$  for an arbitrary scalar  $\Psi$  at the leading order of the expansion in  $\epsilon$ , the equation of motion for  $\Phi$  is given in an extremely simple form as

$$2\mathcal{L}_\lambda \lambda = \mathcal{L}_\lambda \Phi, \quad (5.13)$$

where

$$\mathcal{L}_\lambda := \left( \tilde{\nabla}_\mu \tilde{\nabla}_\nu \Phi + \frac{\Phi}{\ell_\Lambda^2} \tilde{g}_{\mu\nu} \right) \tilde{\mathcal{L}}^{\mu\nu} + \frac{\Delta g^2 - \Delta g_{\mu\nu} \Delta g^{\mu\nu}}{16\Phi^3}. \quad (5.14)$$

We ignore the homogeneous solution  $\mathcal{L}_\lambda \lambda = 0$  by treating  $\mathcal{L}_\lambda^{-1}$  as a local operator as in the case of  $(1 - \hat{H}^2 \hat{\square})^{-1}$  in Sec. 3. The condition that allows such expansion depends on  $\Phi$  and is difficult to express explicitly. Hence, here we just assume that the energy scale is low enough to satisfy this condition. Substituting  $\lambda = \frac{\Phi}{2}$  into Eq. (5.11), we obtain the 4-dimensional effective action from the bulk action as

$$\begin{aligned} S_b &= \frac{M_{pl}^2}{2} \frac{2}{r_c^{(+)}} \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\Delta g^2 - \Delta g_{\mu\nu} \Delta g^{\mu\nu}}{32\Phi} + \frac{\Phi}{2} \left( \tilde{R} - \frac{12}{\ell_\Lambda^2} \right) \right. \\ &\quad \left. - \frac{\Phi}{6} \tilde{\mathcal{L}}^{\mu\nu} \left( (\tilde{\nabla}_\mu \Phi) (\tilde{\nabla}_\nu \Phi) - \frac{\Phi^2}{\ell_\Lambda^2} \tilde{g}_{\mu\nu} \right) \right] + \mathcal{O}(\epsilon^2) \\ &= \frac{M_{pl}^2}{2} \frac{2}{r_c^{(+)}} \int d^4x \sqrt{-g} \left[ \frac{\Delta g^2 - \Delta g_{\mu\nu} \Delta g^{\mu\nu}}{32\Phi} - \frac{1}{2\ell_\Lambda^2} \Phi^2 \left( \square + \frac{4}{\ell_\Lambda^2} \right) \Phi + \frac{\Phi}{2} \left( \tilde{R} - \frac{12}{\ell_\Lambda^2} \right) \right. \\ &\quad \left. - \frac{1}{6} (\nabla_\mu \Phi) (\nabla_\nu \Phi) (\nabla^\mu \nabla^\nu - g^{\mu\nu} \square - R^{\mu\nu}) \Phi \right] + \mathcal{O}(\epsilon^2), \quad (5.15) \end{aligned}$$

where  $\nabla$  is the covariant differentiation associated with  $g_{\mu\nu}$ , which is indistinguishable from  $g_{\mu\nu}^{(\pm)}$  and  $\tilde{g}_{\mu\nu}$  at the leading order of gradient expansion and we use  $R_{\mu\nu} = 3\ell_\Lambda^{-2} g_{\mu\nu} + \mathcal{O}(|K|^2)$  in the second equality. The non-derivative interaction terms between two metrics takes the Fierz-Pauli form  $\Delta g^2 - \Delta g_{\mu\nu} \Delta g^{\mu\nu}$ , which is rewritten in terms of dRGT mass term,  $V_n := \epsilon_{\mu_1 \dots \mu_n}^{\nu_1 \dots \nu_n} \mathcal{U}_{\nu_1}^{\mu_1} \dots \mathcal{U}_{\nu_n}^{\mu_n}$  for  $\mathcal{U}_\nu^\mu := \sqrt{g_{(+)}^{\mu\rho} g_{(-)\rho\nu}}$ , as

$$\Delta g^2 - \Delta g_{\mu\nu} \Delta g^{\mu\nu} = \frac{4}{J} \sum_n c_n V_n. \quad (5.16)$$

Here we define  $J := -\frac{1}{2}(c_1 + 4c_2 + 6c_3)$  and the coefficients  $c_n$  are constrained by the condition that both  $g_{\mu\nu}^{(+)}$  and  $g_{\mu\nu}^{(-)}$  recover Minkowski metric and  $\Delta g_{\mu\nu} \rightarrow 0$  in the limit where the energy densities of matter on the branes and the brane tensions are sent to zero,

$$\begin{aligned} c_4 &= -\frac{1}{24} (c_1 + 6c_2 + 18c_3), \\ c_0 &= -3(c_1 + 2c_2 + 2c_3). \end{aligned} \quad (5.17)$$

We wrote the interaction between two gravitons in terms of dRGT ones, however, we cannot determine the form of mass interactions at higher order of  $\mathcal{U}_\nu^\mu - \delta_\nu^\mu \sim \mathcal{O}(\epsilon)$  at the order of the present approximation. Absorbing  $\Phi\tilde{R}$  term into the induced gravity terms by conformal transformations for  $g_{\mu\nu}^{(\pm)}$ , as in Sec. 3, the action for  $\Phi$  in the second line in Eq. (5.15) is a cubic Galileon [36, 37], and hence the whole action  $S_b + S_+ + S_-$  describes a well-known ghost-free system that contains two interacting metrics and one scalar which couples only to one of the metrics [4]. In order to investigate the coupling of radion as a doubly coupled matter, we have to step into the higher order of the gradient expansion. At the higher order of the gradient expansion, however, higher derivative terms will appear and hence the system may inevitably contain extra degrees of freedom corresponding to the other massive KK gravitons. Therefore, it may not be allowed to obtain doubly coupled matter model using our method, because at the leading order of the gradient expansion the couplings to two metrics is indistinguishable, and the radion may couple to more than two tensor modes if we expand to higher order.

## 6 Summary

In this paper, we intended to obtain the ghost-free bigravity action with a single scalar field from DGP 2-brane model with an unstabilized radion, by solving the bulk equations for given boundary metrics  $g_{\mu\nu}^{(\pm)}$  at  $y = \mp y_0$  and integrating out the bulk degrees of freedom under the approximation that  $g_{\mu\nu}^{(+)} - g_{\mu\nu}^{(-)} \sim |KNy_0| \ll 1$ . As a result, we obtained an action written in terms of  $g_{\mu\nu}^{(\pm)}$  as a 4-dimensional effective theory of DGP 2-brane model, which is reduced to a healthy bigravity model with a scalar field, as expected. Truncating the result at the leading order of the gradient expansion, we obtained the Fierz-Pauli quadratic mass term as the interaction between two gravitons, though we worked on non-linear perturbation. The scalar field couples to only one of the metrics  $g_{\mu\nu}^{(\pm)}$  and its equation of motion does not contain higher-order time derivative. To realize such a setup of the model, we need to tune the brane tension, so that  $|K|$  is sufficiently small. Since this tuning is easily broken by additional matter fields, the energy density of the matter fields on the branes must be small enough.

We succeeded in obtaining the Fierz-Pauli mass interaction naturally from DGP 2-brane model at the lowest order of the gradient expansion. However, it is difficult to extend this method to the higher order of the gradient expansion and to obtain the nonlinear dRGT mass terms, because it will produce complex and higher-derivative interaction terms. Such interaction terms seem to lead extra degrees of freedom in addition to two gravitons and one scalar radion, which will correspond to the appearance of other bulk degrees of freedom. It will be possible to investigate only the higher-order mass interactions between two metrics by taking the limit  $\alpha m_*^2 \rightarrow 0$ , although in this case the mass interaction obtained from the bulk action should be different from the dRGT one. This is because the self-accelerating branch is chosen and radion ghost or Higuchi ghost appears. If ghost appears, the interaction between two metrics  $g_{\mu\nu}^{(\pm)}$  will not take the dRGT form, as is shown in Sec. 3. However, it might be suggestive that the Fierz-Pauli mass term was recovered by fixing the radion by hand in Sec. 3. The investigation of the higher-order mass interactions by considering more extended models is left for future work.

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